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Why is the Solution of the Schrödinger Equation not a Miracle in Quantum Chemistry? A Perspective From the Heisenberg Uncertainty Principle

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ABSTRACT

The Schrödinger equation is the cornerstone of quantum mechanics and is used to determine the energy levels of the hydrogen atom. However, in the teaching of Quantum Chemistry, many students tend to regard the solutions of the Schrödinger equation as purely mathematical results without fully understanding the physical origin of quantum energy levels. In this paper, an approximate model of the hydrogen atom is first developed based on the Heisenberg uncertainty principle and the Coulomb potential. This model is then used to derive the Bohr radius and the ground-state energy by minimizing the total energy function. Subsequently, the time-independent Schrödinger equation is formulated and solved by the method of separation of variables in spherical coordinates to obtain the exact energy spectrum of the hydrogen atom. The results demonstrate that both the approximate model and the exact solution predict the same ground-state energy, $E_1 = -13.6$ eV, while the Schrödinger equation further generalizes this result to the quantized energy spectrum, $E_n = -13.6/n^2$. These findings indicate that the solutions of the Schrödinger equation are not "miraculous" mathematical outcomes but rather the inevitable consequences of the fundamental principles of quantum mechanics. This approach provides a clearer physical interpretation of quantum energy levels and offers significant pedagogical value for the teaching of Quantum Chemistry.

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1. Introduction

The Schrödinger equation is the foundation of quantum mechanics and plays a central role in modern Quantum Chemistry. Solving this equation for the hydrogen atom not only yields the quantized energy levels but also determines the wavefunctions (atomic orbitals), which provide the theoretical basis for explaining atomic structure, chemical bonding, atomic spectra, and many properties of matter at the microscopic scale [1]–[3]. Consequently, the Schrödinger equation has long been a core component of Quantum Chemistry curricula at universities.

However, in current teaching practice, most Quantum Chemistry textbooks begin by introducing the Hamiltonian operator, formulating the Schrödinger equation, and solving it using mathematical techniques such as separation of variables in spherical coordinates, the Legendre equation, and the Laguerre equation [1], [2]. Although this approach is mathematically rigorous, many students, particularly those encountering quantum mechanics for the first time, tend to perceive the energy levels of the hydrogen atom as results that are somehow "generated" by a sequence of sophisticated mathematical manipulations. As a consequence, learners are more likely to focus on the techniques for solving the equation than on the underlying physical principles governing the phenomenon.

In fact, even before solving the Schrödinger equation, the Heisenberg uncertainty principle provides valuable physical insight into the existence of a bound state in the hydrogen atom. By combining the uncertainty principle with the Coulomb potential between the electron and the nucleus, an approximate model of the hydrogen atom can be constructed, from which the Bohr radius and the ground-state energy

are obtained by minimizing the total energy function. Although this model cannot determine the wavefunction or the complete energy spectrum, it accurately predicts the ground-state energy, $E_1 = -13.6$ eV [1], [4], [5]. This demonstrates that quantized energy levels are not arbitrary consequences of mathematical manipulations but rather reflect the fundamental physical laws governing the motion of the electron in the Coulomb field [6], [7].

From this perspective, the Schrödinger equation does not alter the underlying physical picture suggested by the Heisenberg uncertainty principle; rather, it provides a more complete and mathematically rigorous description of the system [8], [9]. Solving the Schrödinger equation not only confirms the ground-state energy but also extends this result to the complete quantized energy spectrum of the hydrogen atom while simultaneously determining the wavefunctions and the quantum numbers that characterize the system [1], [5].

Motivated by the above considerations, this paper presents a two-step approach. First, an approximate model of the hydrogen atom is developed based on the Heisenberg uncertainty principle and the Coulomb potential to determine the Bohr radius and the ground-state energy [10], [11]. Next, the time-independent Schrödinger equation is formulated and solved to obtain the exact solution for the hydrogen atom. By comparing these two approaches, this study demonstrates that the solution of the Schrödinger equation is not a result that appears "miraculously" but rather the mathematical completion of the physical laws already anticipated from the fundamental principles of quantum mechanics. This approach not only provides a deeper understanding of the nature of quantized energy levels but also offers valuable pedagogical implications for improving the teaching of Quantum Chemistry [12].

2. Methods

2.1. Development of an Approximate Model Based on the Heisenberg Uncertainty Principle

The energy eigenvalue of an electron in an atom is the total energy associated with a stationary quantum state of the electron and is equal to the sum of the expectation values of its kinetic and potential energies.

2.1.1. Derivation of the Potential Energy

The electron carries a negative charge, $q_e = -e$, whereas the proton carries a positive charge, $q_p = +e$. The interaction between the electron and the proton is the attractive Coulomb force directed toward the nucleus [1], [2], [7].

$$\overrightarrow{F_c} = \frac{1}{4\pi\varepsilon_o}.\frac{|q_p q_e|}{r^2}.\overrightarrow{e_r} = \frac{1}{4\pi\varepsilon_o}.\frac{e^2}{r^2}.\overrightarrow{e_r} \tag{1}$$

Therefore, the potential energy associated with the attractive Coulomb force is given by [3], [4]:

$$(1) \Rightarrow E_1(r) = -\int_r^\infty \overrightarrow{F_c}.d\vec{r} = -\int_r^\infty \frac{1}{4\pi\varepsilon_o}.\frac{e^2}{r^2}.dr = \left[\frac{1}{4\pi\varepsilon_o}.\frac{e^2}{r}\right]_r^\infty = -\frac{1}{4\pi\varepsilon_o}.\frac{e^2}{r}, \text{ eV} \tag{2}$$

That is,

$$E_1(r) = -\frac{e^2}{4\pi\varepsilon_o.r}, \text{ eV} \tag{3}$$

where $r(m)$ is the distance between the electron and the proton, and $\overrightarrow{e_r}$ is the unit vector directed from the proton to the electron.

2.1.2. Derivation of the Kinetic Energy

According to the Heisenberg uncertainty principle:

$$\Delta x.\Delta p \geq \frac{\hbar}{2} \tag{4}$$

where $\Delta x = x = r/2$; Δp is the uncertainty in the momentum of an electron moving with velocity v (m/s); $\hbar = h/(2\pi)$ is the reduced Planck constant, and $h = 6.625\times10^{-34}$ J.s is the Planck constant [5], [6].

$$(4) \Rightarrow \Delta p \approx \frac{\hbar}{2\Delta x} = \frac{\hbar}{2x} = \frac{\hbar}{r};\ \Delta p \approx m'v \tag{5}$$

where $m' = \dfrac{m_e.m_p}{m_e + m_p}$ (kg) is the reduced mass of the electron–proton system. (6)

Therefore, the kinetic energy of the electron in the atom is given by:

$$E_2(r) = \frac{m'v^2}{2} = \frac{(m'v)^2}{2m'} = \frac{(\Delta p)^2}{2m'} = \frac{\hbar^2}{2m'.r^2} \tag{7}$$

For an electron in an atom, the energy eigenvalue represents the total energy corresponding to a particular stationary quantum state. In quantum mechanics, it is the eigenvalue of the Hamiltonian operator. Before solving the Schrödinger equation, this energy can be estimated using an approximate model based on the Heisenberg uncertainty principle [3]:

$$E(r) = E_1(r) + E_2(r) = -\frac{e^2}{4\pi\varepsilon_o.r} + \frac{\hbar^2}{2m'.r^2} \tag{8}$$

The ground state of the hydrogen atom is the state with the lowest energy. Therefore, the equilibrium radius, r_0 is determined by applying the minimum-energy condition to the total energy function [3], [4]:

$$(8) \Rightarrow \frac{d}{dr}\left(E(r_o)\right) = 0 \Leftrightarrow \frac{e^2}{4\pi\varepsilon_o.r_o^2} - \frac{\hbar^2}{m'.r_o^3} = 0 \tag{9}$$

Solving Equation (9) yields the root: $r_o = \dfrac{4\pi\varepsilon_o.\hbar^2}{m'.e^2}$ (10)

The corresponding result is: $E_{min} = E(r_o) = -\dfrac{m'.e^4}{2(4\pi\varepsilon_o)^2\hbar^2}$ = -13.6 eV (11)

2.2. Formulation and Solution of the Schrödinger Equation for the Hydrogen Atom

2.2.1. Formulation of the Schrödinger Equation

According to the Born interpretation, if the electron is in a normalized quantum state, the total probability of finding the electron throughout all space is equal to unity [2], [8]. That is,

$$\iiint_V |\psi(x,y,z,t)|^2 dV = \iiint_V |\psi|^2 dV = 1 \tag{12}$$

Where $\psi : R^3\times R \rightarrow C$, in the form $\psi = \psi(x,y,z,t)$ is the wave function describing the motion of the electron. According to Schrödinger's formulation of quantum mechanics $\psi = \psi(x,y,z,t)$, the wave function of the hydrogen atom is expressed as:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m'}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V(x,y,z,t).\psi \quad (13)$$

Condense it: $i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m'}\nabla^2\psi+V.\psi$ (14)

Equation (14) is called the Schrödinger equation.

2.2.2. Solution of the Schrödinger Equation

Using the method of separation of variables, let:

$$\psi(x,y,z,t)=\psi(x,y,z).T(t) \quad (15)$$

Substituting Equation (15) into Equation (14) yields:

$$i.\hbar.\psi(x,y,z).\frac{\partial T(t)}{\partial t}=-\frac{\hbar^2}{2m'}T(t).\nabla^2\psi(x,y,z)+V.\psi(x,y,z).T(t) \quad (16)$$

$$(16)\Rightarrow i\hbar\frac{1}{T(t)}\frac{\partial T(t)}{\partial t}=-\frac{\hbar^2}{2m'}\frac{1}{\psi(x,y,z)}.\nabla^2\psi(x,y,z)+V \quad (17)$$

$$(17)\Rightarrow\begin{cases} i\hbar\frac{1}{T(t)}\frac{\partial T(t)}{\partial t}=E \\ -\frac{\hbar^2}{2m'}\frac{1}{\psi(x,y,z)}.\nabla^2\psi(x,y,z)+V=E \end{cases} \quad (18)$$

a) Solution of the Time-Dependent Equation

$$(18)\Rightarrow i\hbar\frac{1}{T(t)}\frac{\partial T(t)}{\partial t}=E\Leftrightarrow\frac{dT(t)}{T(t)}=\frac{-iE}{\hbar}.dt \quad (19)$$

Solving Equation (19) yields the solution:

$$T(t)=A.e^{-\frac{iEt}{\hbar}} \quad (20)$$

b) Solution of the Time-Independent Equation

$$(18)\Rightarrow-\frac{\hbar^2}{2m'}\frac{1}{\psi(x,y,z)}.\nabla^2\psi(x,y,z)+V=E \quad (21)$$

Where E is the total energy (the energy eigenvalue), and V is the potential energy.

$$(21)\Rightarrow-\frac{\hbar^2}{2m'}.\nabla^2\psi+V.\psi=E.\psi \quad (22)$$

Since the hydrogen atom is considered a spherically symmetric system, the wave function $\psi(x,y,z)$ is transformed into spherical coordinates $\psi(r,\theta,\varphi)$ to facilitate the description and solution of Equation (22), by introducing, [2]:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} |J| = r^2.\sin\theta \\ dV = r^2.\sin\theta.dr.d\theta.d\varphi \end{cases} \tag{23}$$

The potential energy is then expressed as:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_o.r}, \text{ eV} \tag{24}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{25}$$

Substituting Equations (24) and (25) into Equation (22) yields:

$$-\frac{\hbar^2}{2m'}\cdot\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] + V.\psi(r,\theta,\varphi) = E.\psi(r,\theta,\varphi) \tag{26}$$

Let: $$\psi(r,\theta,\varphi) = R(r).Y(\theta,\varphi) \tag{27}$$

Substituting Equation (27) into Equation (26) yields the following system of equations, [11], [12]:

$$\begin{cases} L^2Y(\theta,\varphi) = l(l+1)\hbar^2Y(\theta,\varphi) \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m'}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_o.r}\right) - \frac{l(l+1)}{r^2}\right]R = 0 \\ 0 \le r \le +\infty;\ 0 \le \theta \le \pi; 0 \le \varphi \le 2\pi \\ \int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}|\psi|^2.r^2.\sin\theta.dr.d\theta.d\varphi = 1 \end{cases} \tag{28}$$

L^2 is the squared orbital angular momentum operator, which is defined by the following equation:

$$L^2Y_l^m = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2Y_l^m}{\partial\varphi^2}\right] \tag{29}$$

b.1) Solution of the Angular Equation

$$\begin{cases} L^2Y(\theta,\varphi) = l(l+1)\hbar^2Y(\theta,\varphi) \\ 0 \le \theta \le \pi; 0 \le \varphi \le 2\pi \end{cases} \tag{30}$$

The solution of Equation (30) is given by

$$Y(\theta,\varphi) = Y_l^m(\theta,\varphi) = N_{lm}P_l^m(\cos\theta).e^{im\varphi} \tag{31}$$

Where $Y_l^m(\theta,\varphi)$ is the spherical harmonic function; l = 0, 1, 2,… is the orbital quantum number, and $m = -l, -l+1,\ldots, l$ is the magnetic quantum number;

P_l^m is the associated Legendre polynomial, and N_{lm} is the normalization constant, which is defined as follows:

$$P_l^m(x) = (-1)^m \left(1 = x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x);\ \ P_l(x) = \frac{1}{2^l.l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l;\ \ x = \cos\theta \tag{32}$$

$$N_{lm} = (-1)^m \sqrt{\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}} \tag{33}$$

b.2) Solution of the Radial Equation

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[\frac{2m'}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_o.r}\right) - \frac{l(l+1)}{r^2}\right]R = 0 \\ 0 \le r \le +\infty \end{cases} \tag{34}$$

The solution of Equation (34) is given by

$$R(r) = R_{nl}(r) = \left(\frac{2}{nr_o}\right)^{\frac{3}{2}} \sqrt{\frac{(n-l-1)!}{2n(n+l)!}} e^{-\frac{\delta}{2}} .\delta^l L_{n-l-1}^{2l+1}(\delta) \tag{35}$$

Where: $$\delta = \frac{2r}{n.r_o} \tag{36}$$

Substituting Equation (35) into Equation (34) gives the total energy corresponding to a stationary quantum state of the electron, [2]:

$$E_n = -\frac{m'e^4}{2\left(4\pi\varepsilon_o\right)^2\hbar^2} \cdot \frac{1}{n^2} = -\frac{13.6}{n^2},\ \text{eV} \tag{37}$$

Substituting Equations (31) and (35) into Equation (27) gives the following solution:

$$\psi(r,\theta,\varphi) = R_{nl}(r).Y_l^m(\theta,\varphi)$$

$$\psi(r,\theta,\varphi) = \left[\left(\frac{2}{nr_o}\right)^{\frac{3}{2}} \sqrt{\frac{(n-l-1)!}{2n(n+l)!}} e^{-\frac{\delta}{2}} .\delta^l L_{n-l-1}^{2l+1}(\delta)\right] N_{lm}.P_l^m(\cos\theta).e^{im\varphi} \tag{38}$$

Substituting Equations (20) and (38) into Equation (15) yields the general solution of the Schrödinger equation, [2]:

$$\psi(x,y,z,t) = \psi(x,y,z).T(t) \Leftrightarrow \psi(r,\theta,\varphi,t) = \psi(r,\theta,\varphi).T(t)$$

$$\psi(r,\theta,\varphi,t) = \left\{\left[\left(\frac{2}{nr_o}\right)^{\frac{3}{2}} \sqrt{\frac{(n-l-1)!}{2n(n+l)!}} e^{-\frac{\delta}{2}} .\delta^l L_{n-l-1}^{2l+1}(\delta)\right] N_{lm}.P_l^m(\cos\theta).e^{im\varphi}\right\}.A.e^{-\frac{iEt}{\hbar}} \tag{39}$$

3. Results and Discussion

3.1. Simulation of the Approximate Model Based on the Heisenberg Uncertainty Principle

Equation (3) was simulated to obtain the relationship between E_1 and the radius r, illustrating the Coulomb potential well of the electron in the hydrogen atom. The obtained results are shown in Figure 1.

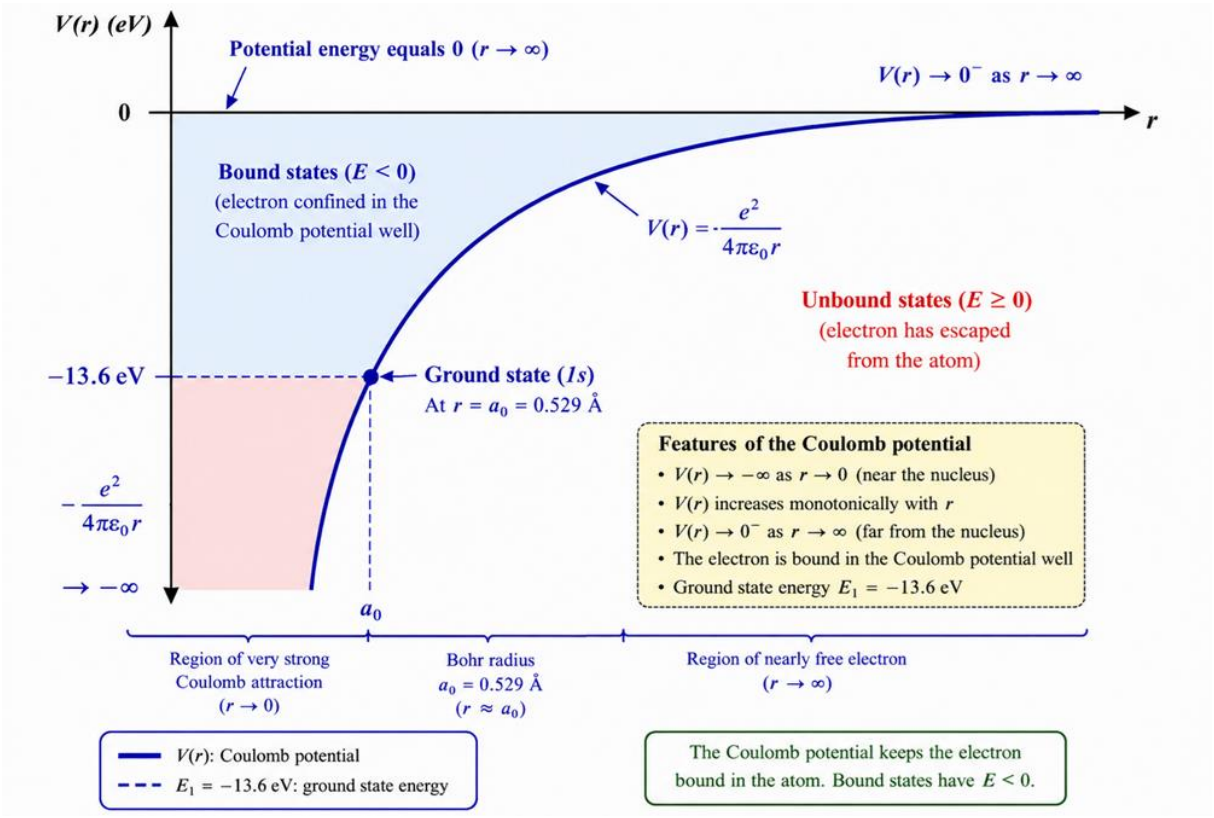

Figure 1. *Coulomb potential well experienced by the electron in the hydrogen atom.*

As shown in Figure 1, when $r < r_0$, the electron is compressed too close to the nucleus, causing the kinetic energy to increase rapidly due to the Heisenberg uncertainty principle. When $r > r_0$, the Coulomb attraction decreases and the potential energy increases. At $r = r_0$, the total energy reaches its minimum value. This corresponds to the Bohr radius of the ground state.

Figure 1 further shows that the the Coulomb potential energy $V(r) = E_1(r) = -\dfrac{e^2}{4\pi\varepsilon_0 . r}$ forms an attractive potential well that confines the electron toward the nucleus. As $r \to 0$, the potential energy decreases rapidly. However, according to the Heisenberg uncertainty principle, the electron cannot be confined indefinitely within the nucleus because the uncertainty in momentum increases significantly, causing the kinetic energy to increase proportionally to $1/r^2$. Conversely, as $r \to \infty$, the Coulomb attractive force gradually decreases, the potential energy approaches zero, and the electron tends to become no longer bound to the nucleus.

The balance between the Coulomb potential energy and the kinetic energy arising from the Heisenberg uncertainty principle produces a minimum point of the total energy function. At this point, the equilibrium radius is determined as $r = a_0 = 0.529$ A^0 ($r_0 = a_0$), and the energy reaches its minimum value of $E = -13.6$ eV, which corresponds to the ground state of the hydrogen atom.

This result has important implications. Based solely on the Coulomb potential energy and the Heisenberg uncertainty principle, the approximate model accurately predicts the Bohr radius and the ground-state energy without solving the Schrödinger equation. This demonstrates that the energy levels do not appear randomly as a consequence of mathematical transformations, but rather originate from the physical nature of the electron–proton system.

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This also provides the foundation for understanding the role of the Schrödinger equation. The equation does not create energy levels "out of nothing"; rather, it provides a complete mathematical description of the quantum state of the electron. When combined with physical boundary conditions, the solutions of the equation not only reproduce the result of the approximate model for the ground state but also determine the entire energy spectrum and atomic orbitals.

3.2. Simulation of the Solution Results of the Schrödinger Equation for the Hydrogen Atom

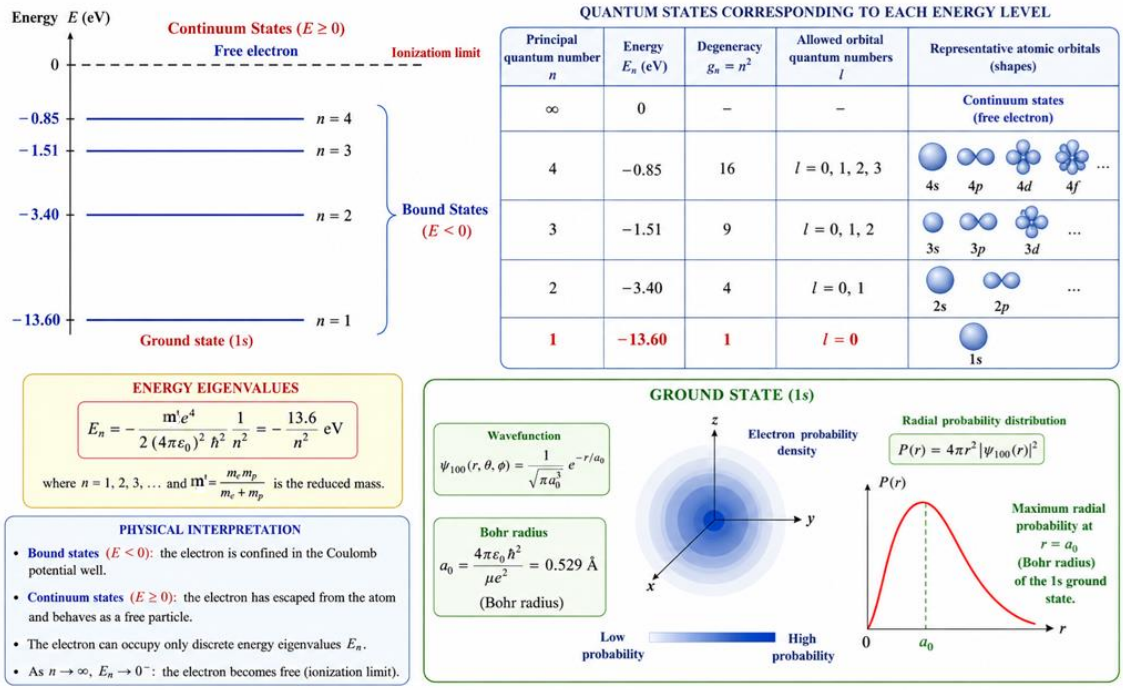

Figure 2. *Schematic diagram of the energy levels of the hydrogen atom.*

The approximate model based on the Heisenberg uncertainty principle can only predict the ground-state energy and the Bohr radius. However, to fully describe the quantum state of the electron, the Schrödinger equation must be solved. In this case, in addition to the discrete energy spectrum, the solutions of the equation also determine the wave functions, atomic orbitals, probability density, and quantum numbers of the electron.

Equation (28), E_n, was simulated as a function of the radius r to describe the energy eigenvalues obtained from solving the Schrödinger equation for the electron in the hydrogen atom. The results are presented in Figure 2.

As shown in Figure 2, the results obtained from the Schrödinger equation are fully consistent with the approximate model based on the Heisenberg uncertainty principle for the ground state. However, the Schrödinger equation also enables the determination of the entire series of excited states, wave functions, and electron probability distributions, which cannot be described by the approximate model.

After applying the separation of variables method and solving the radial equation, angular equation, together with the appropriate physical boundary conditions, the general solution of the Schrödinger equation for the hydrogen atom is obtained as given in Equation (37). The resulting wave function consists of the radial function $R_{nl}(r)$, the spherical harmonic function $Y_l^m(\theta,\varphi)$, and the time-dependent function $T(t)$, providing a complete description of the quantum state of the electron.

From this solution, the energy eigenvalues of the hydrogen atom are determined as $E_n = -13.6/n^2$ eV, where $n = 1, 2, 3,\ldots$ The results show that the electron energy can only take discrete values and depends solely on the principal quantum number n. In particular, when $n = 1$, $E_1 = -13.6$ eV which is in perfect agreement with the result obtained from the approximate model based on the Heisenberg uncertainty principle. This agreement demonstrates that the approximate model correctly captures the physical

nature of the ground state. However, the Schrödinger equation not only determines the energy levels but also provides the complete wave function of the electron. From this wave function, the probability density, shapes of atomic orbitals, quantum numbers, and many other physical quantities can be determined, which cannot be described by the approximate model.

Therefore, the role of the Schrödinger equation is not to "create" energy levels, but rather to establish a complete mathematical model that describes the quantum state of the electron in the Coulomb field. The discrete energy levels emerge as a consequence of solving the equation under appropriate physical boundary conditions.

While the Heisenberg model only predicts the ground-state energy, the Schrödinger equation extends this result to the complete quantum energy spectrum of the hydrogen atom, while also accurately determining the atomic orbitals, probability distributions, and characteristic quantum numbers of the electron.

3.3. Why is the Solution of the Schrödinger Equation Not a Mathematical Miracle?

Based on the results of the approximate model established from the Heisenberg uncertainty principle (Figure 1) and the exact solution of the Schrödinger equation (Figure 2), it is evident that the discrete energy levels of the hydrogen atom do not emerge arbitrarily from mathematical manipulations. Rather, they are inherently determined by the physical nature of the electron–proton system.

According to the Heisenberg uncertainty principle [3], when the electron is confined too close to the nucleus, the uncertainty in position decreases, resulting in a substantial increase in momentum uncertainty. As a consequence, the electron kinetic energy increases approximately as $E_2(r)=\frac{\hbar^2}{2m'.r^2}$, whereas the Coulomb potential energy $V(r)=E_1(r)=-\frac{e^2}{4\pi\varepsilon_o.r}$ decreases with -1/r. The competition between these two energy components leads to a minimum in the total energy function, with $E_{min}=E(r_o)=-\frac{m'.e^4}{2(4\pi\varepsilon_o)^2\hbar^2}$ = –13.6 eV, at the equilibrium radius $r_o=\frac{4\pi\varepsilon_o.\hbar^2}{m'.e^2}$ = 0.529 A^0. This result demonstrates that the electron cannot collapse into the nucleus; instead, it remains localized at a finite distance from the nucleus r_o = 0.529 A^0 (As shown in Figure 3).

Importantly, these quantities can be predicted even before solving the Schrödinger equation, relying solely on fundamental physical principles. This demonstrates that the solution of the Schrödinger equation does not generate the energy levels; instead, it provides an accurate description of the quantum states that are inherently determined by the physical structure of the system.

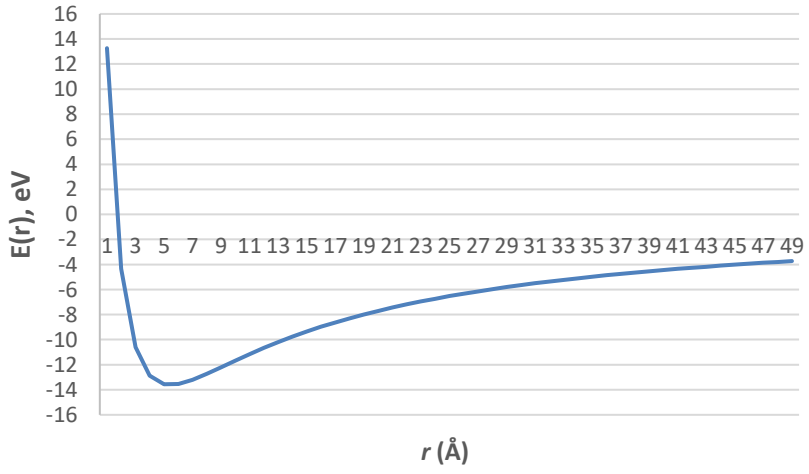

Figure 3. *Graph illustrating the relationship between E(r) and r.*

When solving the Schrödinger Equation [2] under the Coulomb potential with appropriate physical boundary conditions (the wave function remains finite at $r = 0$, is normalizable, and approaches zero as $r \to \infty$), only solutions that satisfy these requirements are physically admissible. These boundary conditions restrict the allowed energy eigenvalues to discrete values $E_n = -\frac{13.6}{n^2}$ eV, rather than a continuous range, where $n = 1, 2, 3, \ldots$. Therefore, energy quantization is not merely a consequence of the mathematical procedure for solving the equation; instead, it is an inevitable outcome of applying the fundamental laws of physics to an electron confined within the Coulomb potential well.

From this viewpoint, the role of the Schrödinger equation must be interpreted properly. The equation does not generate the atomic energy levels; instead, it provides a rigorous mathematical framework for describing the complete quantum state of an electron subjected to a Coulomb potential. When solved with appropriate physical boundary conditions (the wave function must remain finite, continuous, single-valued, and normalizable), the Schrödinger equation not only determines the allowed energy eigenvalues but also yields the complete set of quantum information describing the electron state, including:

- Discrete energy eigenvalues E_n;
- Corresponding energy eigenfunctions $\psi_{nlm}(r, \theta, \phi)$;
- Atomic orbitals and their spatial characteristics;
- Electron probability density and radial probability distributions;
- Quantum numbers n, l, and m that define each quantum state;
- Observable quantities, including expectation values of position, momentum, angular momentum, and energy.

Therefore, the fundamental role of the Schrödinger equation is not to "generate" energy quantization, but to precisely determine the physically allowed quantum states of the electron. Quantization arises naturally as an inevitable consequence of the underlying physical nature of the electron–proton system, which is governed by the Coulomb potential, the Heisenberg uncertainty principle, and the boundary conditions imposed on the wave function. Consequently, the solution of the Schrödinger equation is not a mathematical miracle, but rather a rigorous mathematical manifestation of fundamental physical laws.

4. Conclusions

This study demonstrates that the discrete energy levels of the hydrogen atom do not appear as a "miracle" resulting from the solution of the Schrödinger equation. Even before formulating the Schrödinger equation, the existence of the Bohr equilibrium radius and the approximate ground-state energy of −13.6 eV can be predicted solely from the Coulomb potential and the Heisenberg uncertainty principle. These findings suggest that the characteristic energy scale and the existence of a stable ground state can be anticipated from the Coulomb interaction and the Heisenberg uncertainty principle, providing useful physical insight into the origin of the hydrogen energy spectrum.

When the Schrödinger equation is solved under appropriate physical boundary conditions, the resulting solutions are in excellent agreement with the aforementioned approximate model, while extending a physical estimation into a complete and rigorous mathematical description. The solutions not only determine the energy eigenvalues accurately but also provide the electron wave functions, probability density distributions, atomic orbitals, and the complete set of quantum numbers that characterize the electron's quantum state.

From this viewpoint, the Schrödinger equation can be viewed as the mathematical framework that translates fundamental physical principles and the interaction potential into a quantitative description of quantum states, rather than a mechanism that "generates" energy levels from nothing. Its essential role is to provide a complete mathematical description of the quantum state of a system governed by the potential field and appropriate boundary conditions, whereas the origin of quantization is inherently determined by the physical structure of the system itself.

The findings of this study contribute to a deeper understanding of the relationship between quantum mechanics and quantum chemistry, while offering a more intuitive pedagogical approach. By starting

from the Heisenberg uncertainty principle and subsequently developing toward the Schrödinger equation, learners can first grasp the physical origin of quantum energy levels before engaging with the mathematical formalism. This approach helps prevent the misconception that the solutions of the Schrödinger equation represent a "miracle" of mathematics, and instead highlights them as a natural consequence of fundamental physical principles.

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Conflict of Interest

The authors declare no conflict of interest

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