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A Computational Approach to Examining Dixmier Conjecture in a Specific Case

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ABSTRACT

Dixmier Conjecture on Weyl algebra is one of the central open problems in the field of Lie theory and non-commutative algebra. In this paper, by using the computer algebra system (Maple) we examine a particular instance of this conjecture involving two polynomial generators of relatively low degrees. In parallel, we also study a conjecture introduced in 1997 by Professor Nguyen Huu Anh, which shares deep structural similarities with Dixmier conjecture. Our research reveals a logical relationship between the two conjectures. From a computational perspective, we develop a computer program to systematically construct and analyze all polynomial pairs of degrees (6,9) whose Lie products are constants, thereby confirming the validity of the Dixmier Conjecture in this specific case. Our results contribute to a deeper understanding of the computational and theoretical boundaries of Dixmier conjecture and other related problems in non-commutative algebra.

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1. Introduction

In the seminal work on the Weyl algebra [1], Jacques Dixmier formulated six fundamental problems that have since played a pivotal role in the development of Lie algebra. In 1975, A. Joseph made significant progress by characterizing certain properties of semisimple and nilpotent elements in the Weyl algebra A_1, thereby resolving Problems 3 and 6 (see [2]). Subsequently, Problem 4 concerning homogeneous elements and Problem 5 were studied and resolved in [3].

Among these, the first problem posed by Dixmier asserts that every endomorphism of the Weyl algebra A_1 is an automorphism. The generalization of this statement to higher-dimensional Weyl algebras A_n is now widely known as the Dixmier Conjecture (abbreviated as DC(n)). Despite various efforts and methodological advances over the years [4]–[8], the conjecture remains open even in the base case n = 1.

Notably, the Dixmier Conjecture has been shown to be stably equivalent to the Jacobian Conjecture (JC), a long-standing open problem in algebraic geometry, through a sequence of deep results [9]–[11].

In this paper, we choose to approach Dixmier Conjecture for the case n=1 from a computational perspective. Throughout this paper, denote by k an algebraically closed field of characteristic zero. We identify the Weyl algebra A_1 over k with the associated algebra $k_W[x,y]$ over k generated by two elements x, y satisfying the Lie identity $[x, y] = xy - yx = 1$.

A pair of polynomials (P, Q) is called a *potential polynomial pair* if the Lie bracket $[P, Q]=c\in k$. Each endomorphism of the Weyl algebra A_1 thus can be determined by a potential polynomial pair. Dixmier conjecture $DC_{(1)}$ (named Conjecture D from now on) can thus be restated as follows.

Conjecture D (1965): Every potential polynomial pair (P, Q) in $k_W[x,y]$ whose Lie bracket $[P, Q] = c$ where c is a nonzero constant in k, generates the whole algebra $k_W[x,y]$.

A counterexample to Conjecture D (if exists) is a potential polynomial pair (P, Q) whose Lie bracket $[P,Q] = c \neq 0$ that does not generate the whole algebra $k_W[x,y]$, i.e. $k_W[x,y] \subsetneq k_W[x,y]$.

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Let Λ be the set of all potential polynomial pairs (P, Q) of degrees (6,9) in the Weyl algebra $\mathbb{C}_W[x, y]$. In this paper, one of our main objectives is to describe all potential polynomial pairs (P, Q) of degrees (6,9). To this end, we leverage the computational capabilities of the software Maple to develop algorithms for computing Lie brackets of polynomials in $\mathbb{C}_W[x, y]$, as well as for solving a specific class of systems of polynomial equations. The source code of our program is stored at [12] to facilitate independent verification if necessary. To the best of our knowledge, no such algorithm has been previously reported. The calculations in our program's output led us to obtain a computer-based proof of the following result.

Theorem 1.1 (Theorem 3.3). *Let (P,Q) be a polynomial pair in the set Λ. Then $[P,Q]=0$.*

A consequence of this Theorem is that there is no counterexample to Dixmier conjecture which is a pair of polynomials of degrees (6, 9). Note that this was already proved in the work [7]. However, along with advances in computer science and the rapid increase in computational speed, our approach is expected to be able to investigate the potential polynomial pairs (P, Q) of higher degrees where $\gcd(\deg P, \deg Q) > 15$ in near future. The work of computing the Lie braket of two general polynomials (P,Q) and forcing $[P,Q]=c$ involves a huge number of computations beyond the capabilities of standard computers. To solve this problem, we first prove the following proposition, which enables one restricts to the case of polynomials (P, Q) where $\gcd(\deg P, \deg Q) < \min(\deg P, \deg Q)$ when exploring possible counterexample to Dixmier conjecture.

Proposition 1.2 (Ref. to Proposition 2.4). *In the Weyl algebra $k_W[x, y]$, if there exists a polynomial pair (M,N) such that*

- $[M, N] = 1$
- $k_W[M,N] \subsetneq k_W[x, y]$ *and* $\gcd(\deg M, \deg N) = \min(\deg M, \deg N)$

then there exists a polynomial pair (P_1, Q_1) such that

- $[P_1, Q_1] = 1$
- $k_W[P_1, Q_1] \subsetneq k_W[x, y]$ *and* $\gcd(\deg P_1, \deg Q_1) < \min(\deg P_1, \deg Q_1)$.

Secondly, in the case where $\gcd(\deg P, \deg Q)=3$, we prove the following result.

Theorem 1.3 (See Theorem 3.2). *Let P, Q be polynomials in $\mathbb{C}_W[x, y]$ of degrees p, q where* $\gcd(p,q)=3<\min(p,q)$. *Suppose that $[P,Q]=c \in \mathbb{C}$ and P,Q have the form $P = u_0^{p/3} + P_1, \deg(P_1) < p$, $Q = u_0^{q/3} + Q_1, \deg(Q_1) < q$, where $u_0 = x^3 + ax^2y + bxy^2 + cy^3 \in \mathbb{C}[x, y]$. Then, there are polynomials P', Q' in $\mathbb{C}_W[x, y]$ of the form $P' = u^{\frac{p}{3}} + P'_1, \deg(P'_1) < p$, $Q' = u^{\frac{q}{3}} + Q'_1, \deg(Q'_1) < q$, such that $[P', Q']=c \in \mathbb{C}$ and the polynomial u takes one of three forms: $u = x^3$, $u = x^2y$, or $u = x(x^2 - y^2)$.*

Based on this theorem, the highest homogeneous parts of polynomials (P, Q) are reduced to three cases described above, so the number of calculations is significantly reduced. Consequently, our algorithms run successfully, which allows us to exhaustively describe polynomial pairs (P, Q) of degrees (6, 9) where $[P, Q]$ is a constant.

Another goal in this paper is to investigate a conjecture, named Conjecture A, which was proposed by Professor Nguyen Huu Anh after conducting numerous Lie brackets of various polynomial pairs when working at ICTP (Italy) in 1996-1997.

Conjecture A: Assume that M, N are two polynomials in $k_W[x, y]$ of degrees m, n where $m \geq 2$ or $n \geq 2$, $\gcd(m,n) < \min(m,n)$, and also the Lie product $[M, N]=c$ for some constant $c \in k$. Then, there exists a polynomial $\alpha = \alpha(x, y)$ with degree $d = \gcd(m,n)$ and two univariate polynomials F, $G \in k[X]$ such that $M(x, y) = F(\alpha(x, y))$ and $N(x, y) = G(\alpha(x, y))$. Therefore $[M,N] = 0$.

At one hand, we show that Conjecture A is stronger than Conjecture D, in the sense that the validity of Conjecture A implies the validity of Conjecture D (see Theorem 2.5), on the other hand we find a counterexample to Conjecture A.

Finally, in this article, we find a counterexample to Conjecture A (see Theorem 3.4), and we exhaustively explore potential polynomial pairs of degrees (9,5), as consequence we can verify the

Dixmier Conjecture in the case of polynomial pairs of degrees (6,9) by a computer-assisted approach. As a continuation of this work, our forthcoming objective is to enhance the current algorithms to facilitate a more comprehensive exploration of potential polynomial pairs of higher degrees.

2. Some properties of polynomial in the Weyl algebra A_1

We identified the Weyl algebra $A_1(k)$ with

$$k_W[x,y] := \left\{ \sum_{i,j} a_{ij}\, x^i y^j, \text{ where } a_{ij} \in k,\ [x,y] = xy - yx = 1 \right\} \tag{1}$$

The group $\mathrm{Aut}(k_W[x,y])$ contains the types of elementary automorphisms:

1. Linear automorphisms: $\omega_1 : (x,y) \to (ax+by,\ cx+dy)$, where $ad - bc \neq 0$
2. Triangular automorphisms $\omega_2 : (x,y) \to (x, \varphi(x)+y)$, where $\varphi(x) \in k[x]$.

Dixmier ([1], 1965) proved the following result on the automorphism group $\mathrm{Aut}(A_1)$:

Theorem 2.1. *If char $k = 0$, the automorphism group* $\mathrm{Aut}(A_1)$ *of the Weyl algebra* $A_1(k)$ *generated by elementary automorphisms ω_1, ω_2 defined as above.*

Definition 2.2. In the non-commutative (Weyl) algebra $k_W[x,y]$, a polynomial pair (P,Q) is called a *potential polynomial pair* if its Lie product $[P,Q]=$ c for some $c \in k$.

A counter example of Dixmier's conjecture (if exists) is a potential polynomial pair (P,Q) in $k_W[x,y]$ satisfying $[P,Q]=1$ and $k_W[P,Q] \subsetneq k_W[x,y]$. To date, there is no proof of the correctness of Dixmier conjecture, so a possible approach is to find a counterexample for it. The following result is well known.

Lemma 2.3. *Given two homogeneous polynomials $M=M(x,y)$, $N=N(x,y)$ of degrees $m,n \geq 1$ in the commutative polynomial algebra $k[x,y]$. Assume that the determinant of the Jacobian matrix $\partial(M,N)/\partial(x,y)$ is zero. Then there exists a homogeneous polynomial $\alpha \in k[x,y]$ of degree $u = \gcd(m,n)$ such that $M = c_1 \alpha^{m/u}$, $N = c_2 \alpha^{n/u}$ for some $c_1, c_2 \in k$.*

We prove the following result which shows that there is a surjective map from the set C of counterexamples of Dixmier conjecture to the set $C^<$ of counterexamples (P,Q) of Dixmier conjecture where $\gcd(\deg P, \deg Q) < \min(\deg P, \deg Q)$.

Proposition 2.4. *In the Weyl algebra $k_W[x,y]$, if there exists a polynomial pair (M,N) such that*

- $[M,N] = 1$
- $k_W[M,N] \subsetneq k_W[x,y]$ *and* $\gcd(\deg M, \deg N) = \min(\deg M, \deg N)$

then there exists a polynomial pair (P_1, Q_1) such that

- $[P_1, Q_1] = 1$
- $k_W[P_1, Q_1] \subsetneq k_W[x,y]$ *and* $\gcd(\deg P_1, \deg Q_1) < \min(\deg P_1, \deg Q_1)$.

In other words, if there exists (M,N) which is a counterexample of conjecture D with $\gcd(\deg M, \deg N) = \min(\deg M, \deg N)$, *then we can find another counterexample (P_1, Q_1) such that* $\gcd(\deg P_1, \deg Q_1) < \min(\deg P_1, \deg Q_1)$.

Proof. Let $m = \deg M$ and $n = \deg N$. We consider the two following cases.

1. Case $m = n = 1$: it is obvious.
2. Case $\max(m,n) > 1$: First we assume $m<n$, so $\gcd(m,n) = \min(m,n) = m$. Denote u_m, v_n respectively be the homogeneous parts of degrees m,n *in* M, N. Denote by H the homogeneous part of degree $m+n-2$ of the bracket $[u_m, v_n]$, then clearly $H = \det\left(\frac{\partial(u_m, v_m)}{\partial(x,y)}\right)$. Since $[M,N]=1$, $H=0$. By Lemma 2.3, there exists a homogeneous polynomial $\alpha(x,y)$ of degree $m = \gcd(m,n)$ such that $u_m = \alpha(x,y)$ and $v_n = \alpha(x,y)^{n/m}$. Write $M = u_m + M^-$ and $N = u^{n/m} + N^-$ where M^-, N^- are polynomials of degrees strictly

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less than m, n respectively. Let $Q_1 = N - M^{n/m}$. Obviously, $[M, Q_1] = [M, N] = 1$ and the pair (M, Q_1) is obtained from polynomials M, N by an elementary transformation of type 1, as a result $k_W[M, Q_1] = k_W[M, N] \subsetneq k_W[x, y]$. Hence, the proof is completed by induction on the degrees of potential polynomial pairs.

Theorem 2.5. *Assume that conjecture A holds true. Then conjecture D holds true as well.*

Proof. Suppose that conjecture A is true. If conjecture D is false, then there exists a Lie algebra automorphism $\omega : k_W[x, y] \to k_W[x, y]$ which is not surjective. Let $M = \omega(x)$, $N = \omega(y)$, then $[M, N] = 1$ and (M, N) do not generate the Weyl algebra $k_W[x, y]$. Denote by m, n the degrees of polynomials M, N, then

- If $\gcd(m, n) < \min(m, n)$, then conjecture A implies that $[M, N] = 0$. This is absurd.
- If $\gcd(m, n) = \min(m, n)$, Proposition 2.4 asserts that there exist polynomials P_1, Q_1 with degree p_1, q_1 such that $[P_1, Q_1] = 1$ and $\gcd(p_1, q_1) < \min(p_1, q_1)$. So $[P_1, Q_1] = 0$ due to conjecture A. This is a contradiction. Thus, there is no counterexample to conjecture D.

3. Exploring potential polynomial pairs (*M*, *N*) of degrees (6,9)

3.1. Verifying Conjecture D for polynomial pairs of degrees (6,9)

Lemma 3.1. *Given two polynomials* $P, Q \in k_W[x, y]$ *with* $\deg P = p > 1$ *and* $\deg Q = q > 1$. *Then the total degree of the Lie bracket* $[P, Q]$ *is bounded as* $\deg([P, Q]) \le p + q - 2$.

Proof. It is sufficient to prove this lemma for the case $P = x^{m_1}y^{n_1}, Q = x^{m_2}y^{n_2}$ and this case is done by Lemma 2.1 of [1].

Now, let M and N be polynomials as follows:

$$M = \sum_{n=0}^{6} \sum_{i+j=n;\ i,j\ge 0} A_{ij}x^i y^j, \quad N = \sum_{m=0}^{9} \sum_{i+j=m;\ i,j\ge 0} B_{ij}x^i y^j \tag{2}$$

By Lemma 3.1, we can rewrite Lie bracket of M and N in (1) as a polynomial in $k_W[x, y]$ of degree $6 + 9 - 2 = 13$,

$$[M, N] = f_{00} + \sum_{n=1}^{13} \sum_{s+t=n;\ s,t\ge 0} f_{st}x^s y^t \tag{3}$$

where $f_{st} \in \mathbb{Z}[U]$ are polynomials with integer coefficients on the set of unknowns $U = \{A_{ij} |\, i + j \le 6\} \cup \{B_{ij} |\, i + j \le 9\}$.

We write a computer program using the computational algebra software Maple to describe all potential polynomial pairs (M, N) of degrees (6, 9). Our source code can be found at [12].

The constraint $[M, N] = c$ with $c \in k$ leads to solving the system of equations

$$f_{st} = 0, \text{for } 1 \le s + t \le 13 \tag{4}$$

The system (4) consists of 14 + 13 + · · · 2 = 104 non-linear polynomial equations and 83 unknowns from the above set U. It is almost impossible to solve this system by hand without the aid of a computer. First, we write

$$M = P^+ + P_1, \deg(P_1) < 6, \text{and} \quad N = Q^+ + Q_1, \deg(Q_1) < 9 \tag{5}$$

where P^+ is the homogeneous part of degree 6 in M and Q^+ is the homogeneous part of degree 9 in N. Then the condition $[M, N] = c \in k$ follows that $\det\left[\frac{\partial(P^+, Q^+)}{\partial(x,y)}\right] = 0$, so by Lemma 2.3 we can rewrite

$$M = u_0^2 + P_1, \deg(P_1) < 6, \text{and} \quad N = u_0^3 + Q_1, \deg(Q_1) < 9 \tag{6}$$

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where polynomial $u_0 = x^3 + ax^2y + bxy^2 + cy^3 \in k[x,y]$.

We prove the following theorem which is crucial in reducing the number of unknowns in the set *U*.

Theorem 3.2. *Let P, Q be polynomials in* $\mathbb{C}_W[x,y]$ *of degrees* p, q *satisfying the condition* $\gcd(p,q) = 3 < \min(p,q)$. *Suppose that [P, Q]=c* ∈ ℂ *and that P, Q have the form*

$$P = u_0^{p/3} + P_1, \deg(P_1) < p, \quad Q = u_0^{q/3} + Q_1, \deg(Q_1) < q \tag{7}$$

where $u_0 = x^3 + ax^2y + bxy^2 + cy^3 \in \mathbb{C}[x,y]$. *Then, there are polynomials P′, Q′* ∈ $\mathbb{C}_W[x,y]$ *of the form*

$$P' = u^{p/3} + P'_1, \deg(P'_1) < p, \quad Q' = u^{q/3} + Q'_1, \deg(Q'_1) < q \tag{8}$$

such that [*P′*, *Q′*] = *c* ∈ℂ *and the polynomial u takes one of the forms*

$$u = x^3, u = x^2y, or\ u = x(x^2 - y^2) \tag{9}$$

Proof. Now let $t = x/y$ and $f(t) = t^3 + at^2 + bt + c = u_0/y^3$, then $f(t)$ has three roots t_0, t_1, t_2 in the algebraic closed field *k* = ℂ. We consider three following cases.

i. If $t_0 = t_1 = t_2$ then $f(t) = (t - t_0)^3$ and so $u_0 = (t - t_0)^3 y^3 = (x - t_0 y)^3$. By applying triangular automorphism $\omega : (x,y) \to (x + t_0 y, y)$ in $\mathrm{Aut}(k_W[x,y])$, we obtain that

$$P' = \omega(P) = u^{p/3} + P'_1, \deg(P'_1) < p, \quad Q' = \omega(Q) = u^{q/3} + Q'_1, \deg(Q'_1) < q \tag{10}$$

in which $u = \omega(u_0) = x^3$, $P'_1 = \omega(P_1)$, $Q'_1 = \omega(Q_1)$, as a result the Lie bracket $[P', Q'] = [\omega(P), \omega(Q)] = c$ as desired.

ii. If $t_0 = t_1 \neq t_2$ then $u_0 = (x - t_0 y)^2 (x - t_2 y)$. Similarly, by the automorphism

$$\omega : (x,y) \to \left(\frac{t_2 x - t_0 y}{t_2 - t_0}, \frac{x - y}{t_2 - t_0}\right), \text{ we then obtain } u = \omega(u_0) = x^2 y. \tag{11}$$

iii. If t_0, t_1, t_2 are all distinct then $u_0 = (x - t_0 y)(x - t_1 y)(x - t_2 y)$, explicitly

$$u_0 = (x - t_0 y)\,[x^2 - (t_1 + t_2)xy + t_1 t_2 y^2] \tag{12}$$

We make change of the variables by letting $X = \alpha(x + t_0 y)$, and $Y = mx - ny$ for some numbers *α*, *m*, *n* ∈ ℂ. We have $X^2 - Y^2 = (\alpha^2 - m^2)x^2 - 2(\alpha^2 t_0 + mn)xy + (\alpha^2 t_0^2 - n^2)y^2$. One can easily find complex scalars *α*, *m*, *n* such that

$$u_0 = \frac{1}{\alpha} X(X^2 - Y^2) \tag{13}$$

If $\alpha \neq 0$, by similar argument as above, applying appropriate automorphism $\omega \in \mathrm{Aut}(k_W[x,y])$ we get that $\omega(u_0) = \frac{1}{\alpha} x(x^2 - y^2)$. Again, by applying the automorphism $\omega' : (x,y) \to (\sqrt[3]{\alpha}x, \sqrt[3]{\alpha}y)$ we obtain that $u = \omega'(\omega(u)) = x(x^2 - y^2)$ as desired. Finally, note that if $\alpha = 0$ or equivalently $m^2 = -1$ then t_1, t_2 must satisfy an interesting relation $-t_1 t_2 = (t_1 + t_2)^2/4$. This special relation can be broken easily by making change $t \longmapsto t + \epsilon$ for some appropriate $\epsilon \in \mathbb{C}$ (apply $\omega : (x,y) \to (x + ey, y)$). Hence it is sufficient to treat the case $\alpha \neq 0$, which is already done.

In implementation, we input into our Maple program [12] the polynomial pair

$$P = u^2 + \text{terms of degrees} \leq 5, \text{and } Q = u^3 + \text{ terms of degrees} \leq 8 \tag{14}$$

For the three cases $u = x^3$, $u = x^2y$, $u = x(x^2 - y^2)$, our program runs successfully in solving the equation [*P*, *Q*] = *c* where *c* ∈ ℂ. In each case, our program outputs a list of all potential polynomials. It turns out that all their Lie brackets are zero. So, we obtain:

Theorem 3.3. *Let* (*P*, *Q*) *be a polynomial pair in the set* Λ. Then [*P*, *Q*] = 0.

3.2. Finding counterexamples for Conjecture A

Theorem 3.4. *Let*

$$\begin{aligned} Q(x,y) &= x^9 + 3x^6y^2 + 3x^3y^4 + y^6 + 3x^4 + 3y^2x + 3y \\ P(x,y) &= x^6 + 2x^3y^2 + y^4 - 6x^2y + 2x \end{aligned} \tag{15}$$

Then $[P, Q] = 0$, *and the pair* (P, Q) *cannot be simultaneously expressed as univariate polynomials in* u *where* $u = u(x,y) = x^3 + ay^2 + bxy + cx^2 + dx + ey$ *for some* $a, c, b, d \in \mathbb{C}$.

Proof. Assume that there are univariate polynomials $F(X) = X^2 + a_1X + a_2$, $G(X) = X^3 + b_1X^2 + b_2X + b_3$ both in the univariate ring $\mathbb{C}[X]$, and a cubic polynomial $u = x^3 + ay^2 + bxy + cx^2 + dx + ey \in \mathbb{C}[x,y]$ such that

$$\begin{aligned} P(x,y) &= F(u) = u^2 + a_1u + a_2, \quad a_i \in \mathbb{C} \\ Q(x,y) &= G(u) = u^3 + b_1u^2 + b_2u + b_3, \quad b_i \in \mathbb{C} \end{aligned} \tag{16}$$

By a simple computation we see that $b = c = d = e = 0$ and $a = 1$, so $u = x^3 + y^2$. Then we have $P(x,y) = u^2 + 2x$, $Q(x,y) = u^3 + 3ux + 3y$. Thus, P and Q cannot be expressed as polynomials in only u. To finish the proof, we check $[P, Q] = 0$.

$$[P,Q] = [u^2 + 2x, u^3 + 3ux + 3y] = 6[x, ux + y] + 2[x, u^3] + 3[u^2, ux + y] \tag{17}$$

Note that $[x,u] = [x,y^2] = 2y$; $[u,y] = [x^3,y] = 3x^2$. It is also easy to see that

$$6[x, ux + y] = 6([x, ux] + 1) = 6([x,u]x + 1) = 6(2yx + 1) = 6(2xy - 1) \tag{18}$$

On the other hand, by straightforward computations, we obtain that

$$\begin{aligned} 2[x,u^3] + 3[u^2, ux + y] &= 2u[x,u^2] + 2[x,u]u^2 + 3u[u^2,x] + 3[u^2,y] \\ &= [u,[u,y]] = [u, 3x^2] = 3[y^2, x^2] = -12xy + 6 \end{aligned} \tag{19}$$

Hence, we see that $[P, Q] = 0$.

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Conflict of Interest

The author declares no conflict of interest.

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